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Theoretical Foundation of the Equations for the Generation of Surface Coordinates

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Nomenclature†

$b_{\alpha\beta}$	$=\boldsymbol{n}\cdot\boldsymbol{r}_{,\alpha\beta}$; coefficients of the second fundamental form
D	= second-order differential operator; Eq. (2)
G_ν	$=g_{\alpha\alpha}g_{\beta\beta}-(g_{\alpha\beta})^2$, (α,β,ν) cyclic, e.g., $G_3=g_{11}g_{22}-(g_{12})^2$
$g_{\alpha\beta}$	$=\boldsymbol{r}_{,\alpha}\cdot\boldsymbol{r}_{,\beta}$; covariant metric components
$g^{\alpha\beta}$	= contravariant metric components
$g^{\alpha\sigma}g_{\beta\sigma}$	$=\delta^\alpha_\beta$; thus, $g^{11}=g_{22}/G_3$, $g^{12}=-g_{12}/G_3$, $g^{22}=g_{11}/G_3$
k_I, k_{II}	= principal curvatures at a point in the surface
$\boldsymbol{n}$	$=(\boldsymbol{r}_{,\alpha}\times\boldsymbol{r}_{,\beta})/(\sqrt{G_\nu})$; unit normal vector on the surface ($x^\nu=$ const, e.g., $\nu=3$: $\alpha=1$, $\beta=2$)
$\boldsymbol{r}$	$=(x_i)$; $i=1, 2, 3$, the Cartesian coordinates
R	= defined in Eq. (4)
x^α	= two dimensional curvilinear coordinates; (α, β, ν) cyclic
x^ν	= const, defines the surface; $\nu=1,2,3$
$\Gamma^\delta_{\alpha\beta}$	$=\frac{1}{2}g^{\sigma\delta}(g_{\alpha\sigma,\beta}+g_{\beta\sigma,\alpha}-g_{\alpha\beta,\sigma})$; surface Christoffel symbols of the second kind
Δ_2	$=\dfrac{1}{\sqrt{G_3}}\dfrac{\partial}{\partial x^\alpha}\left((\sqrt{G_3})g^{\alpha\beta}\dfrac{\partial}{\partial x^\beta}\right)$, Beltramian/Laplacian Operator

Introduction

DESPITE the availability of a large body of literature on elliptic grid generation, there has not been much discussion on the foundational aspects of the model equations which are used to generate the grid lines. It always appears that the equations, called the grid generators, have been chosen on an ad hoc basis except when the grids are to be generated in the two-dimensional physical x-y space. In the later case early authors (e.g., Winslow,[1] Chu,[2] and Thompson et al.[3]) chose the Laplace equations as the grid generators based on their simplicity and above all because of the existence of a maximum principle for such equations. Brackbill,[4] in search of a criteria for grid smoothness on the basis of a variational principle, again established that the Laplace equations form the most optimum grid generators for the two-dimensional physical x-y space. It is when one tries to develop a set of elliptic grid generators for a curved surface in the three-dimensional physical x, y, z space that the whole question on the foundational aspect again comes forward. Warsi[5-8] has proposed a set of elliptic partial differential equations that are based on some simple differential-geometric results. In essence, the starting point of the works in Refs. 5-8 is the fact that the coordinates to be generated in a surface must satisfy the equations of Gauss, viz., Eq. (A1). (Refer to the Appendix for some essential formulas.) Besides the equations of Gauss, the surface grid equations must also satisfy the equations of Weingarten, viz., Eq. (A2). It is the purpose of this Note to show that the model proposed in Refs. 5-8 also satisfies the equations of Weingarten. With the satisfaction of the preceding equations, the proposed model equations for the surface have a fundamental basis and thus form an optimum set of grid generators.

Model for Coordinate Generation

In this section we first use only Eq. (A1) (method I) and obtain the coordinate generation equations. Next, we use only Eq. (A2) (method II) and again obtain the same equations.

Method I

Inner multiplication of Eq. (A1) by $g^{\alpha\beta}$ and the use of the results

$$\Delta_2 x^\delta = -g^{\alpha\beta}\Gamma^\delta_{\alpha\beta}, \qquad k_I + k_{II} = g^{\alpha\beta}b_{\alpha\beta}$$

yields the equation

$$g^{\alpha\beta}\boldsymbol{r}_{,\alpha\beta} + (\Delta_2 x^\delta)\boldsymbol{r}_{,\delta} = \boldsymbol{n}(k_I + k_{II}) \tag{1}$$

Writing $x^1=\xi$, $x^2=\eta$, introducing the operator D

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} \tag{2}$$

and $P=\Delta_2\xi$, $Q=\Delta_2\eta$, Eq. (1) becomes

$$D\boldsymbol{r} + G_3(P\boldsymbol{r}_\xi + Q\boldsymbol{r}_\eta) = \boldsymbol{n}R \tag{3}$$

where

$$R = G_3(k_I + k_{II}) = g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22} \tag{4}$$

Equation (3) provides three scalar equations for the determination of x, y, z. To obtain a deterministic set of equations from Eq. (3), we have to prescribe the Beltramians $\Delta_2\xi$, $\Delta_2\eta$.

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†Greek indicates (except ν) assume two values. A comma before a suffix, say α, indicates a partial derivative with respect to the curvilinear coordinate x^α. Thus, $\boldsymbol{r}_{,\alpha}=\delta\boldsymbol{r}/\delta x^\alpha$, $\boldsymbol{r}_{,\alpha\beta}=\delta^2\boldsymbol{r}/\delta x^\alpha\delta x^\beta$. On the other hand, a variable subscript denotes a partial derivative, e.g., $\boldsymbol{r}_\xi=\delta\boldsymbol{r}/\delta\xi$, $\boldsymbol{r}_\nu=\delta\boldsymbol{r}/\delta\nu$, etc. Summation convention is implied when the same index appears both as a lower and an upper index.

For details on this aspect and also on the specification of $k_I + k_{II}$, refer specifically to Ref. 7. It may be noted that, if the surface degenerates into a plane, then Eq. (3) reduces to the inverted form of a set of Poisson equations.

Method II

First of all, the Beltramian operator Δ_2 as defined in the Nomenclature is used to write

$$\Delta_2 r = \frac{1}{\sqrt{G_3}} \frac{\partial}{\partial x^\alpha} (\sqrt{G_3} g^{\alpha\beta} r_{,\beta}) \tag{5}$$

Opening the differentiation and using the expression for $\Delta_2 x^\delta$, we get

$$\Delta_2 r = g^{\alpha\beta} r_{,\alpha\beta} + (\Delta_2 x^\delta) r_{,\delta} \tag{6}$$

which is the left-hand side of Eq. (1). We now use the identity

$$g^{\alpha\beta} r_{,\beta} = \epsilon^{\alpha\delta} r_{,\delta} \times n$$

where

$$\epsilon^{11} = 0, \quad \epsilon^{12} = 1/\sqrt{G_3}, \quad \epsilon^{21} = -1/\sqrt{G_3}, \quad \epsilon^{22} = 0$$

in Eq. (5) and get

$$\Delta_2 r = \frac{1}{\sqrt{G_3}} (r_{,2} \times n_{,1} - r_{,1} \times n_{,2}) \tag{7}$$

Using the Weingarten's equation (A2) in Eq. (7), we get

$$\Delta_2 r = b_{\alpha\beta} g^{\alpha\beta} n$$
$$= n(k_I + k_{II}) \tag{8}$$

Equating the right-hand-side of Eqs. (6) and(8), we obtain Eq. (1). This demonstration shows that the proposed model, Eq. (3), satisfies both the Gauss and the Weingarten equations.

As has been noted earlier, Eq. (3) provides three scalar equations for the Cartesian coordinates. It has been shown by Thompson et al.[9] and Warsi[10] that from Eq. (3) two parametric equations can also be obtained by a simple application of the chain rule. Let u and v be the parametric curves in the given surface in which the curvilinear coordinates ξ and η are to be generated. Writing

$$\bar{g}_{11} = r_u \cdot r_u, \quad \bar{g}_{12} = r_u \cdot r_v, \quad \bar{g}_{22} = r_v \cdot r_v$$

$$\bar{G}_3 = \bar{g}_{11}\bar{g}_{22} - (\bar{g}_{12})^2, \qquad J_3 = u_\xi v_\eta - u_\eta v_\xi$$

then from the expressions such as

$$r_\xi = r_u u_\xi + r_v v_\xi, \qquad r_\eta = r_u u_\eta + r_v v_\eta, \text{ etc.}$$

and simple algebraic manipulations, one gets from Eq. (3)

$$a u_{\xi\xi} - 2b u_{\xi\eta} + c u_{\eta\eta} + J_3^2 (P u_\xi + Q u_\eta) = J_3^2 \overline{\Delta_2} u \tag{9}$$

$$a v_{\xi\xi} - 2b v_{\xi\eta} + c v_{\eta\eta} + J_3^2 (P v_\xi + Q v_\eta) = J_3^2 \overline{\Delta_2} v \tag{10}$$

where

$$a = g_{22}/\bar{G}_3, \quad b = g_{12}/\bar{G}_3, \quad c = g_{11}/\bar{G}_3$$

and

$$\overline{\Delta_2} u = \frac{1}{\sqrt{\bar{G}_3}} \left[\frac{\delta}{\delta u} \left(-\frac{\bar{g}_{22}}{\sqrt{\bar{G}_3}} \right) - \frac{\delta}{\delta v} \left(\frac{\bar{g}_{12}}{\sqrt{\bar{G}_3}} \right) \right]$$

$$\overline{\Delta_2} v = \frac{1}{\sqrt{\bar{G}_3}} \left[\frac{\delta}{\delta v} \left(\frac{\bar{g}_{11}}{\sqrt{\bar{G}_3}} \right) - \frac{\delta}{\delta u} \left(\frac{\bar{g}_{12}}{\sqrt{\bar{G}_3}} \right) \right]$$

Equations (9) and (10) were also obtained independently by Garon and Camarero.[11] For extensive computational results using Eqs. (3), (9), and (10), refer to Refs. 12 and 13.

Conclusions

This Note demonstrates that any solution of the set of Eqs. (3) will simultaneously satisfy both the equations of Gauss and Weingarten. Thus, Eq. (3) is not chosen on an ad hoc basis but has its foundations in some fundamental results of diffential geometry.

Appendix: Theoretical Foundation

The basic partial differential equations of the surface theory (e.g., refer to Struik,[14] Willmore,[15] Kreyszig,[16] and Warsi[17]) are given in the following. To fix ideas, we consider the surface $\nu = 3$ or $x^3 = \text{const}$ so that all of the Greek indices in the following equations assume values 1, 2 only.

The equations (or formulas) of Gauss:

$$r_{,\alpha\beta} = \Upsilon^\delta_{\alpha\beta} r_{,\delta} + n b_{\alpha\beta}; \qquad \alpha, \beta = 1,2 \tag{A1}$$

The equations of Weingarten:

$$n_{,\alpha} = -g^{\beta\gamma} b_{\alpha\beta} r_{,\gamma}; \qquad \alpha = 1,2 \tag{A2}$$

The equations of Gauss define the coordinates $r = (x_i)$ of a surface as functions of x^1 and x^2. These equations have to satisfy certain compatibility conditions that are obtained from

$$(r_{,\alpha\beta})_{,\gamma} = (r_{,\alpha\gamma})_{,\beta} \tag{A3}$$

which yield two vector equations; one for $\alpha = 1$, $\beta = 1$, $\gamma = 2$ and the other for $\alpha = 2$, $\beta = 2$, $\gamma = 1$. Using Eqs. (A1) and (A2) in Eq. (A3), we obtain

$$b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = \Upsilon^\epsilon_{\alpha\gamma} b_{\epsilon\beta} - \Upsilon^\epsilon_{\alpha\beta} b_{\epsilon\gamma} \tag{A4}$$

which are two equations: one for $\alpha = 1$, $\beta = 1$, $\gamma = 2$ and the other for $\alpha = 2$, $\beta = 2$, $\gamma = 1$. These equations are known as the Codazzi equations. The other outcome of Eq. (A3) is the Gauss equation

$$R^\delta_{,\alpha\gamma\beta} = g^{\mu\delta} (b_{\alpha\beta} b_{\gamma\mu} - b_{\alpha\gamma} b_{\beta\mu}) \tag{A5}$$

where

$$R^\delta_{,\alpha\gamma\beta} = \Upsilon^\delta_{\alpha\beta,\gamma} - \Upsilon^\delta_{\alpha\gamma,\beta} + \Upsilon^\epsilon_{\alpha\beta}\Upsilon^\delta_{\epsilon\gamma} - \Upsilon^\epsilon_{\alpha\gamma}\Upsilon^\delta_{\epsilon\beta}$$

is the Riemann-Christoffel tensor. Four distinct equations from Eq. (A5) can be obtained by taking

$$\delta = 1: \quad \alpha = 1, \quad \beta = 2, \quad \gamma = 1; \quad \alpha = 2, \quad \beta = 2, \quad \gamma = 1$$

$$\delta = 2: \quad \alpha = 1, \quad \beta = 2, \quad \gamma = 1; \quad \alpha = 2, \quad \beta = 2, \quad \gamma = 1$$

The covariant curvature tensor is defined as

$$R_{\tau\alpha\gamma\beta} = g_{\delta\tau} R^\delta_{,\alpha\gamma\beta}$$

The only distinct component of the covariant curvature tensor in surface theory is

$$R_{1212} = b_{11} b_{22} - (b_{12})^2$$

from which the Gaussian curvature K is

$$K = R_{1212}/G_3$$

The fundamental theorem of the surface theory states that "If $g_{\alpha\beta}$ and $b_{\alpha\beta}$ are sufficiently differentiable *given* functions of x^1 and x^2, which satisfy Eqs. (A4) and (A5) while $G_3 \neq 0$, then there exists a surface whose first and second fundamental forms are, respectively,

$$I = g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad II = b_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

This surface is uniquely determined except for its position in space. The construction of the surface, viz., the determination of the Cartesian coordinates x_i ($i=1,2,3$), then proceeds by solving the fifteen scalar equations obtained from Eqs. (A1) and (A2) under the conditions

$$n \cdot n = 1, \quad n \cdot r_{,\delta} = 0 (\delta = 1,2), \quad r_{,\alpha} \cdot r_{,\beta} = g_{\alpha\beta}$$

$$n \cdot r_{,\alpha\beta} = b_{\alpha\beta}, \qquad \alpha = 1,2; \qquad \beta = 1,2$$

However, it must be noted that in comparison to the aims of surface theory the aim of grid generation is to generate lines in a *given* surface. Despite the difference in aims, the basic equations of Gauss and Weingarten must always be satisfied.

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Reduction of Component Mode Synthesis Formulated Matrices for Correlation Studies

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Introduction

MANY of the published techniques for improving an analytical model or identifying areas of discrepancy in the model using measured test information requires reduction of the mass and/or stiffness matrices of the structure to the same degrees of freedom as measured during testing.[1-3] For structural systems that have a large number of degrees of freedom or have components designed by separate groups or organizations the method of component mode synthesis has proven to be an accurate, efficient, and economical method of analysis. The technique of component mode synthesis introduces generalized variables or Ritz coefficients that are not measurable or derivable quantities from testing into the final solution set. During a modal test of the actual structure, only physical variables are measured; therefore, it would be highly desirable to have a method that would operate directly on the final component mode synthesis system matrices by the elimination of generalized variables and any additional physical degrees of freedom not measured during testing while maintaining the same validity of the original system matrices. In this Note a technique for accomplishing this objective is described and subsequently applied to a numerical example.

Theory

The final system equations from a component mode synthesis formulation using constraint modes may be partitioned into measured physical variables U_a, nonmeasured physical variables U_o, and generalized variables U_q:

$$\begin{bmatrix} M_{aa} & M_{ao} & M_{aq} \\ M_{oa} & M_{oo} & M_{oq} \\ M_{qa} & M_{qo} & M_{qq} \end{bmatrix} \begin{Bmatrix} \ddot{U}_a \\ \ddot{U}_o \\ \ddot{U}_q \end{Bmatrix} + \begin{bmatrix} K_{aa} & K_{ao} & K_{aq} \\ K_{oa} & K_{oo} & K_{oq} \\ K_{qa} & K_{qo} & K_{qq} \end{bmatrix} \begin{Bmatrix} U_a \\ U_o \\ U_q \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

The nonmeasured physical and generalized variables may be combined into a single vector of deleted variables U_d:

$$U_d = \begin{Bmatrix} U_o \\ U_q \end{Bmatrix} \tag{2}$$

Therefore, Eq. (1) may be rewritten in the following form:

$$\begin{bmatrix} M_{aa} & M_{ad} \\ M_{da} & M_{dd} \end{bmatrix} \begin{Bmatrix} \ddot{U}_a \\ \ddot{U}_d \end{Bmatrix} + \begin{bmatrix} K_{aa} & K_{ad} \\ K_{da} & K_{dd} \end{bmatrix} \begin{Bmatrix} U_a \\ U_d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

where the entries in the mass and stiffness matrices are corresponding partitions of Eq. (1).

It is desired to express the U_d solution variables, which are not measured during testing, in terms of the U_a solution vari-

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